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Symmetries of the three-body problem

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Abstract. The symmetries of a three-body problem, e.g. representing a model of three quarks, are determined. The relevant definitions and theorems concerning symmetries, like the symmetry condition, are listed. The software that helps solve the symmetry condition by computer is discussed.

1. Introduction

In an extremely simplified version of QCD it is assumed that the gluon field holding the three quarks in a proton together can be approximated by the action of three strings in the way shown in figure 1. The energy of the gluon field is chosen to be proportional to the length of the string and the midpoint t is defined by the requirement that $u_1 + u_2 + u_3$ be a minimum.

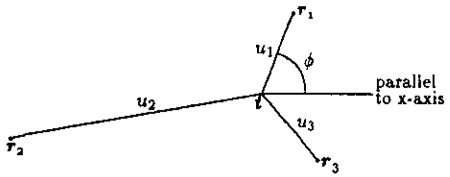

Figure 1. Three strings holding three quarks together

The plane in which the motion takes place is regarded as a part of $\mathbb{R}^2$ and we denote the point r_i by the vector $\begin{pmatrix} x_i \\ y_i \end{pmatrix}$. The Lagrangian associated to this problem is given by [1]

$$L = \tfrac{1}{2}m(|\dot{r}_1|^2 + |\dot{r}_2|^2 + |\dot{r}_3|^2) - 3mkR \tag{1.1}$$

where

$$R = \tfrac{1}{3}\min_{s}\sum_{i=1}^{3}|r_i - s| = \tfrac{1}{3}(u_1 + u_2 + u_3) \tag{1.2}$$

$$m = \text{the mass of each particle} \tag{1.3}$$

$$k = \text{a coupling constant} \tag{1.4}$$

$$r_i = \text{the position of particle } i \qquad (i = 1, 2, 3) \tag{1.5}$$

$$u_i = \text{the distance between } r_i \text{ and t}. \tag{1.6}$$

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We assume each of the angles of the triangle with points $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ and $\boldsymbol{r}_3$ to be less than 120°. This assumption results, as Torricelli has already proved, in angles of 120° between the strings $\boldsymbol{r}_i - \boldsymbol{t}$, $i = 1, 2, 3$.

In this paper the symmetries of this three-body problem are determined. The power of the computer for the study of symmetries and related problems is demonstrated. Symmetries are important for obtaining information regarding the solutions of a system of (partial) differential equations. An algorithm to determine symmetries of a differential equation is given by Olver [2]. All computations made in this paper are done by using the software developed by Kersten [3] and Gragert [4]. In the second section we describe the relevant aspects for obtaining symmetries of a system of differential equations. The algebraic point of view of a differential equation is explained, and the notion of a symmetry is defined and the symmetry condition is given. The third section is devoted to a short description of the software used to derive and to solve the linear first order system of differential equations that results from the symmetry condition. In the final section we compute symmetries of the Euler–Lagrange equations derived from the variational problem (1.1).

2. The symmetry condition

Lie was the first to discover the importance of symmetries in differential equations. Appropriate references can be found in [2].

Several authors (e. g. Ibragimov [5], Olver [2], Ovsiannikov [6] and Vinogradov [7]) discussed the various aspects of symmetries of differential equations. Here we will follow Olver's work (*Applications of Lie groups to Differential Equations*) and all references in this section are, unless stated otherwise, with respect to this work. We consider an n^{th} order system of differential equations Δ involving p independent variables $x = (x^1, \ldots, x^p)$ and q dependent variables $u = (u^1, \ldots, u^q)$. Multi-indices $I = (i_1, \ldots, i_p)$, $|I| = i_1 + \cdots + i_p$, enable a compact notation for the derivatives of u. The variable u^j_I corresponds to the derivative of u^j with respect to x; e.g. ($u^2_{(1,2,0)} = \partial^3 u^2/\partial x_1 \partial x_2^2$) and ($u^5_{(3,1,0,1)} = \partial^5 u^5/\partial x_1^3 \partial x_2 \partial x_4$), or $u^2_{210} = u^2_{(2,1,0)}$ and $u^5_{3101} = u^5_{(3,1,0,1)}$ if no confusion is possible.

The space J^k given by $\{(x, u_I)_{|I| \leqslant k}\}$ is called the k^{th} order jet space. Formally, we also have the *inifinite jet space* J^∞ that contains all derivatives of u. Functions of J^∞ are functions defined of some J^k with k finite, i.e. they depend on a finite number of variables. The n^{th} order system Δ can be seen as an algebraic equation $F = 0$ on the n^{th} order jet space J^n, where F is defined on J^n.

Example. The Euler–Lagrange equations of (1.1) are given by

$$\ddot{x}_i = -3k(\partial R/\partial x_i) \qquad \ddot{y}_i = -3k(\partial R/\partial y_i) \qquad i = 1, 2, 3 \tag{2.1}$$

where R, given by (1.2), is a function of x_1, y_1, x_2, y_2, x_3 and y_3. This is a second-order system of differential equations that involves one independent variable $x^1 = t$ and six dependent variables $u^1 = x_1$, $u^2 = y_1$, $u^3 = x_2$, $u^4 = y_2$, $u^5 = x_3$ and $u^6 = y_3$. The differential equations (2.1) can be seen as a system of algebraic equations on the *second order jet space*

$$J^2 = \{(x^1, u^1, \ldots, u^6, u^1_1, \ldots, u^6_1, u^2_1, \ldots, u^6_2)\}. \tag{2.2}$$

The algebraic equations are given by

$$u_2^j + 3k\frac{\partial R}{\partial u^j} = 0 \qquad j = 1,\ldots,6 \tag{2.3}$$

where R is a function of $u^1,\ldots,u^6$.

Here we are interested in continuous groups of point symmetries of the system Δ. These are one-parameter groups of transformations, i.e. flows, on J^0 such that solutions are transformed to solutions of Δ (p 96, Def 2.23). Functions are transformed by the transformation of their graphs in J^0. A flow on J^0 induces a flow on J^k such that the derivatives of the function $u = f(x)$ are transformed into the derivatives of the transformed function. The induced flow on J^k is called the kth order prolongation of the flow on J^0. A flow on J^0 is a symmetry of Δ if and only if it leaves the differential equation invariant (p 103).

For the concept of a symmetry the correspondence between a flow and a vector field is important. A vector field can be seen as the infinitesimal generator of a flow and integration of the vector field gives back the flow. The general expression for a vector field on J^0 is given by

$$v = \sum_{i=1}^{p}\xi^i\frac{\partial}{\partial x^i} + \sum_{j=1}^{q}\phi^j\frac{\partial}{\partial u^j} \tag{2.4}$$

where ξ^i and ϕ^j are functions defined on J^0.

The prolongation of a vector field on J^0 to a vector field on J^k is given by the infinitesimal version of the prolongation of the associated flow. In order to describe this prolongation of the vector field the formal total derivatives on J^∞ are introduced. The formal total derivative with respect to x^m is given by

$$\mathcal{D}_m = \frac{\partial}{\partial x^m} + \sum_{j=1}^{q}\sum_{I} u^j_{I,m}\frac{\partial}{\partial u^j_I}, \qquad m = 1,\ldots,p \tag{2.5}$$

where the multi-index I,m in $u^j_{I,m}$ is given by $(i_1,\ldots,i_m+1,\ldots,i_p)$. There is no convergence problem in taking the infinite sum over I in $\mathcal{D}_m$, since $\mathcal{D}_m$ acts only on functions defined on some finite order jet space. We also define $\mathcal{D}^I$ to be $(\mathcal{D}_1)^{i_1}\circ\cdots\circ(\mathcal{D}_p)^{i_p}$, where $(\mathcal{D}_m)^j = (\mathcal{D}_m)^{j-1}\circ\mathcal{D}_m$.

Theorem (p 113, Th 2.36). Prolongation of vector fields.

The k^{th} order prolongation of v (2.4) is given by

$$\mathrm{pr}^k v = v + \sum_{j=1}^{q}\sum_{|I|=1}^{k}\phi_I^j\frac{\partial}{\partial u^j_I} \qquad \phi^j_I = \mathcal{D}^I\Big(\phi^j - \sum_{i=1}^{p}\xi^i u^j_{e_i}\Big) + \sum_{i=1}^{p}\xi^i u^j_{I,i} \tag{2.6}$$

where e_i is the multi-index with 1 as its i^{th} index and the other indices are 0. A vector field on J^0 is called an infinitesimal symmetry if its corresponding flow is a symmetry.

Theorem (p 106-7, Th 2.31). Symmetry condition.

A vector field v on J^0 is an infinitesimal symmetry of the n^{th} order system Δ given by $F = 0$ if and only if

$$\mathrm{pr}^n v(F) = 0 \qquad \text{whenever} \quad F = 0\,. \tag{2.7}$$

This theorem states that the symmetry condition results in a system of linear differential equations for the coefficients of the vector field v, i.e. ξ^i and ϕ^j in (2.4). This system (2.7) is, besides linear, also overdetermined, because the unknown functions ξ^i and ϕ^j depend only on J^0, whereas the system depends on u_I with $1 \leqslant |I| \leqslant n$. From now on, we make no distinction between infinitesimal symmetries and their associated symmetries. This identification is permitted by the unambiguous connection between vector fields and flows.

The symmetries of Δ generate a Lie algebra, with the usual Lie bracket for vector fields. Symmetries of the Euler–Lagrange equations can give rise to conservation laws. The correspondence is established by the variational symmetries of the variational problem.

A one-parameter group of transformations, i.e. a flow, is called a variational symmetry of a variational problem given by the lagrangian L, if the quantity

$$\mathcal{L}[f] = \int_X L(x, f(x))\mathrm{d}x \tag{2.8}$$

does not change by transformation of the function and the area of integration.

Theorem (p 257, Th 4.12). A vector field is a variational symmetry of (2.8) if and only if

$$\mathrm{pr}^n v(L) + L\sum_{i=1}^{p} \mathcal{D}_i \xi^i = 0\,. \tag{2.9}$$

Variational symmetries are symmetries of the Euler–Lagrange equations (p 259, Th 4.14), but the converse is not true.

It was Noether who established the one-to-one correspondence between variational symmetries of a variational problem and conservation laws of its Euler–Lagrange equations (p 278, Th 4.29). In case the Lagrangian is defined on J^1 the conservation law corresponding to the variational symmetry (2.4) is given by (p 279, Corollary 4.30)

$$\sum_{i=1}^{p} \mathcal{D}_i \left(\sum_{j=1}^{q} \phi_j \frac{\partial L}{\partial u^j_{e_i}} + \xi^i L - \sum_{j=1}^{q}\sum_{k=1}^{p} \xi^k u^j_{e_k} \frac{\partial L}{\partial u^j_{e_i}} \right) = 0 \qquad \text{when} \quad E(L) = 0\,. \tag{2.10}$$

A natural generalization of the notion of point symmetries is established by the notion of the so-called generalized symmetries, also known as higher order symmetries. Generalized symmetries arise by allowing the infinitesimal generator of the flow, i.e. the coefficients of the vector field (2.4), to depend also on derivatives of the dependent variables. According to Olver, it appears that for non-linear equations an infinite series of generalized symmetries *is related to complete integrability*. Vinogradov [7] has set up the theory of coverings in which the notion of non-local symmetries, an extension of the notion of generalized symmetry, is aesthetically justified. Here we will determine the point symmetries of the Euler–Lagrange equations (2.1). We have shown that point symmetries arise as the solution of an overdetermined linear system of differential equations, which enable a computer algebraic approach. In the next section we will show how to use the computer to derive and to solve the symmetry condition.

3. Software

In the previous section we have seen that the symmetry condition is an overdetermined linear system of differential equations, reflecting the prolongation of a vector field and the invariance of the differential equation by the action of the prolonged vector field. The computations involved are quite mechanical, and therefore well suited to be performed by a computer using symbolic algebra.

Here we will explain the software for symmetry computations made by Kersten [3], for that is what we will use in the next section to determine the symmetries of the Euler–Lagrange equations (2.1). Kersten uses the differential geometry package in REDUCE developed by Gragert [4], which enables computations with vector fields, differential forms, wedge products and Lie derivatives. Other symbolic manipulation programs that enable symmetry computations have been developed by Champagne and Winternitz [8] and Schwarz [9].

Kersten [3] approaches the symmetry condition from the differential form side, as do Harrison and Estabrook [10]. We shall not go into the aspects that are raised by their method and only remark that the result, i.e. the symmetry condition, is the same. We describe the overdetermined linear system that is constructed by the computer and the methods that are essential to solve this system and which have therefore been implemented in the program.

Consider the n^{th} order differential equation Δ corresponding to $F = 0$, where F is defined on J^n. The overdetermined linear system is represented by a set of differential equations for the coefficients of a vector field $\boldsymbol{v}^n$ on J^n

$$\boldsymbol{v}^n = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{j=1}^{q} \sum_{|I| \leqslant n} \phi_I^j \frac{\partial}{\partial u_I^j} . \tag{3.1}$$

The set of differential equations is given by equations the solution of which arrange $\boldsymbol{v}^n$ to be prolonged, i.e. $\boldsymbol{v}^n = \mathrm{pr}^n \boldsymbol{v}$ for some vector field $\boldsymbol{v}$ on J^0,

$$\phi_{I,k}^j - \left(\mathcal{D}_k(\phi_I^j) - \sum_{i=1}^{p} (\mathcal{D}_k \xi^i) u_{I,i}^j \right) = 0 \qquad |I| \leqslant n-1 \quad k = 1, \ldots, p \tag{3.2}$$

and equations, the solution of which implies the action of $\boldsymbol{v}^n$ on F to vanish

$$\left(\sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{j=1}^{q} \sum_{|I| \leqslant n} \phi_I^j \frac{\partial}{\partial u_I^j} \right) F = 0 . \tag{3.3}$$

Once $F = 0$ has been solved with respect to highest order variable(s), the equations (3.2) and (3.3) constitute an overdetermined linear system. The unknown functions of this system, ξ^i and ϕ_I^j, are implemented as being represented by functions F^i, $i \geqslant 1$.

The methods that are essential to solve the system are programmed to be checked one by one for their execution and if possible realised with respect to an equation in the overdetermined linear system. Each method will be accompanied by an example that originates with a system that has three variables, x, y and z, and the dependency of the functions in the system is given by $F^1 = F^1(x, y)$ and $F^2 = F^2(x)$.

1. If the equation is polynomial with respect to a variable, then the equation is broken up with respect to this variable, i.e. new equations are considered by demanding the coefficients of the polynomial to vanish. In this way the whole polynomial behaviour is treated.

Example: Equation $z^2 \frac{\partial F^1}{\partial y} + zF^2 = 0$

Solution $\frac{\partial F^1}{\partial y} = 0$ and $F^2 = 0$

2. If the equation is just a derivative (may be of higher order) of a function, then the equation is integrated, i.e. the function is specified in functions that depend on one less variable.

Example: Equation $\frac{\partial^2 F^1}{\partial x \partial y} = 0$

Solution $F^1 = F^3 + F^4$ where $F^3 = F^3(x)$ and $F^4 = F^4(y)$

3. If the equation can be solved with respect to a function, taking into account the depencies, then this is executed.

Example: Equation $xF^1 + y\frac{\partial F^2}{\partial x} = 0$

Solution $F^1 = -y\frac{\partial F^2}{\partial x} \Big/ x$

4. If the equation contains a term that is the product of a number and the derivative of a function, e.g. F^1, with respect to variables that are not present as arguments of functions in the other terms of the equation then the equation is integrated.

Example: Equation $\frac{\partial F^1}{\partial y} + xF^2 = 0$

Solution $F^1 = -xyF^2 + F^3$ where $F^3 = F^3(x)$

5. If the equation contains a function that depends on a variable that does not appear as an argument of functions in the remaining part of the equation, then an appropriate differential consequence can lead to an equation which already has been treated and its solution then induces polynomial behaviour of the original equation.

Example: Equation $x\frac{\partial F^1}{\partial x} + yF^2 = 0$

Differential consequence $x\frac{\partial^3 F^1}{\partial x \partial y^2} = 0$

Solution $F^1 = F^3 + yF^4 + F^5$ where $F^3 = F^3(y)$

Solution $F^4 = F^4(x)$ and $F^5 = F^5(x)$

Equation $xy\frac{\partial F^4}{\partial x} + x\frac{\partial F^5}{\partial x} + yF^2 = 0$

Solution $x\frac{\partial F^4}{\partial x} + F^2 = 0$ and $x\frac{\partial F^5}{\partial x} = 0$

These five methods and additional considerations due to the specific nature of the problem at hand are in many practical problems adequate to solve the overdetermined linear system.

4. Symmetries of the three-body problem

In the first section the three-body problem that we consider has been described. It is a variational problem given by the Lagrangian (1.1)

$$L = \tfrac{1}{2}m(|\dot{\boldsymbol{r}}_1|^2 + |\dot{\boldsymbol{r}}_2|^2 + |\dot{\boldsymbol{r}}_3|^2) - 3mkR \tag{4.1}$$

where

$$R = R(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3) = \tfrac{1}{3}\min_{\boldsymbol{s}} \sum_{i=1}^{3} |\boldsymbol{r}_i - \boldsymbol{s}| . \tag{4.2}$$

In this section we apply the software described in the previous section to compute the symmetries of the associated Euler–Lagrange equations (2.1)

$$\ddot{x}_i = -3k(\partial R/\partial x_i) \qquad \text{and} \quad \ddot{y}_i = -3k(\partial R/\partial y_i) \quad i = 1, 2, 3 . \tag{4.3}$$

The angles of the triangle originated by the points $\boldsymbol{r}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$, $i = 1, 2, 3$ are assumed to be less than 120° (cf section 1).

Lemma. We will show that R (4.2) is in this case given by

$$R = \frac{1}{3}\Big(|\boldsymbol{r}_1|^2 + |\boldsymbol{r}_2|^2 + |\boldsymbol{r}_3|^2 - (\boldsymbol{r}_1 \cdot \boldsymbol{r}_2 + \boldsymbol{r}_2 \cdot \boldsymbol{r}_3 + \boldsymbol{r}_3 \cdot \boldsymbol{r}_1) \\ + \sqrt{3}|\det(\boldsymbol{r}_1, \boldsymbol{r}_2) + \det(\boldsymbol{r}_2, \boldsymbol{r}_3) + \det(\boldsymbol{r}_3, \boldsymbol{r}_1)|\Big)^{1/2} \tag{4.4}$$

where '$\cdot$' denotes the inner product and $\det(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is the determinant $\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$.

Proof. Geometrical considerations imply invariance with respect to translation and rotation and scaling of R

$$\begin{aligned} R(\boldsymbol{r}_1 - \boldsymbol{a}, \boldsymbol{r}_2 - \boldsymbol{a}, \boldsymbol{r}_3 - \boldsymbol{a}) &= R(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3) \qquad \boldsymbol{a} \in \mathbb{R}^2 \\ R(M_\alpha \boldsymbol{r}_1, M_\alpha \boldsymbol{r}_2, M_\alpha \boldsymbol{r}_3) &= R(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3) \qquad M_\alpha \text{ is rotation over angle } \alpha \\ R(\lambda \boldsymbol{r}_1, \lambda \boldsymbol{r}_2, \lambda \boldsymbol{r}_3) &= \lambda R(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3) \qquad \lambda > 0 \end{aligned} \tag{4.5}$$

Therefore it suffices to prove formula (4.4) in the case $\boldsymbol{r}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\boldsymbol{r}_2 = a\begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2}\sqrt{3} \end{pmatrix}$ and $\boldsymbol{r}_3 = b\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2}\sqrt{3} \end{pmatrix}$. In figure 1.1 these three points are established by transformation over $-\boldsymbol{t}$, rotation over $-\phi$ and scaling by $1/u_1$, for Torricelli proved the angles between the strings $\boldsymbol{r}_i - \boldsymbol{t}$, $i = 1, 2, 3$, to be 120°.

Finally,

$$R\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, a\begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2}\sqrt{3} \end{pmatrix}, b\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2}\sqrt{3} \end{pmatrix}\right) = \frac{1 + a + b}{3}$$

as it should be. □

The algebraic approach, as outlined in the second section, of the Euler–Lagrange equations (4.3) is obtained by considering t, x_1, y_1, x_2, y_2, x_3, y_3, $\dot{x}_1$, $\dot{y}_1$, $\dot{x}_2$, $\dot{y}_2$, $\dot{x}_3$, $\dot{y}_3$, $\ddot{x}_1$, $\ddot{y}_1$, $\ddot{x}_2$, $\ddot{y}_2$, $\ddot{x}_3$, $\ddot{y}_3$ as variables of the second order jet space J^2.

Symmetries of the Euler–Lagrange equations (4.3) are represented by vector fields on $J_0 = \{(t, x_1, y_1, x_2, y_2, x_3, y_3)\}$

$$v = F^1\frac{\partial}{\partial t} + F^2\frac{\partial}{\partial x_1} + F^3\frac{\partial}{\partial y_1} + F^4\frac{\partial}{\partial x_2} + F^5\frac{\partial}{\partial y_2} + F^6\frac{\partial}{\partial x_3} + F^7\frac{\partial}{\partial y_3} \tag{4.6}$$

where $F^1, \ldots, F^7$, functions on J^0, are solutions of the symmetry condition. As we have seen in the previous section, the symmetry condition is given by the overdetermined linear system for the coefficients of a vector field on J^2

$$\begin{aligned} v^2 = v &+ F^8\frac{\partial}{\partial \dot{x}_1} + F^9\frac{\partial}{\partial \dot{y}_1} + F^{10}\frac{\partial}{\partial \dot{x}_2} + F^{11}\frac{\partial}{\partial \dot{y}_2} + F^{12}\frac{\partial}{\partial \dot{x}_3} + F^{13}\frac{\partial}{\partial \dot{y}_3} \\ &+ F^{14}\frac{\partial}{\partial \ddot{x}_1} + F^{15}\frac{\partial}{\partial \ddot{y}_1} + F^{16}\frac{\partial}{\partial \ddot{x}_2} + F^{17}\frac{\partial}{\partial \ddot{y}_2} + F^{18}\frac{\partial}{\partial \ddot{x}_3} + F^{19}\frac{\partial}{\partial \ddot{y}_3} \end{aligned} \tag{4.7}$$

where $F^8, \ldots, F^{13}$ are defined on J^1 and $F^{14}, \ldots, F^{19}$ are defined on J^2. This system consists of 'prolongation' equations, i.e. their solution results in $v^2 = \mathrm{pr}^2 v$, and of 'invariance' equations, i.e. their solution results in $v^2(F) = 0$ whenever $F = 0$.

The algebraic Euler–Lagrange equation $F = 0$ is solved with respect to the variables that represent second order derivatives and therefore the overdetermined linear system is defined on J^1.

The overdetermined linear system ([11], appendix c) appears to have already solved the prolongation equations for the coefficients $F^{14}, \ldots, F^{19}$. The first six equations of the overdetermined linear system express the coefficients $F^8, \ldots, F^{13}$ by prolongation in the coefficients of the vector field v, i.e. $F^1, \ldots, F^7$ and each of these first six equations look like

$$\begin{aligned} \mathrm{Equ}^1 = {} & F^2_t + \dot{x}_1 F^2_{x_1} + \dot{x}_2 F^2_{x_2} + \dot{x}_3 F^2_{x_3} + \dot{y}_1 F^2_{y_1} + \dot{y}_2 F^2_{y_2} + \dot{y}_3 F^2_{y_3} \\ & - \dot{x}_1 F^1_t - \dot{x}_1^2 F^1_{x_1} - \dot{x}_1\dot{x}_2 F^1_{x_2} - \dot{x}_1\dot{x}_3 F^1_{x_3} \\ & - \dot{x}_1\dot{y}_1 F^1_{y_1} - \dot{x}_1\dot{y}_2 F^1_{y_2} - \dot{x}_1\dot{y}_3 F^1_{y_3} - F^8 . \end{aligned} \tag{4.8}$$

The subscript variables denote derivatives with respect to those variables and $\mathrm{Equ}^1 = 0$ defines the first equation. The remaining six equations of the overdetermined linear system induce the desired action of the vector field v^2. An impression of such an equation is given by

$$\begin{aligned} \mathrm{Equ}^7 = {} & F^8_t + \dot{x}_1 F^8_{x_1} + \dot{x}_2 F^8_{x_2} + \dot{x}_3 F^8_{x_3} - 3kR_{x_1} F^8_{\dot{x}_1} - 3kR_{x_2} F^8_{\dot{x}_2} \\ & - 3kR_{x_3} F^8_{\dot{x}_3} + \dot{y}_1 F^8_{y_1} + \dot{y}_2 F^8_{y_2} + \dot{y}_3 F^8_{y_3} - 3kR_{y_1} F^8_{\dot{y}_1} \\ & - 3kR_{y_2} F^8_{\dot{y}_2} - 3kR_{y_3} F^8_{\dot{y}_3} + 3kR_{x_1} F^1_t + 3k\dot{x}_1 R_{x_1} F^1_{x_1} \\ & + 3k\dot{x}_2 R_{x_1} F^1_{x_2} + 3k\dot{x}_3 R_{x_1} F^1_{x_3} + 3k\dot{y}_1 R_{x_1} F^1_{y_1} + 3k\dot{y}_2 R_{x_1} F^1_{y_2} \\ & + 3k\dot{y}_3 R_{x_1} F^1_{y_3} + 3kR_{x_1x_2} F^3 + 3kR_{x_1x_3} F^4 + 3kR_{x_1y_1} F^5 \\ & + 3kR_{x_1y_2} F^6 + 3kR_{x_1y_3} F^7 + 3kR_{x_1x_1} F^2 . \end{aligned} \tag{4.9}$$

Equ[7] clearly shows that the formula for R has not yet been used, only the dependencies have been specified, preventing the system being prematurely big. The first six equations contain the exact behaviour of all the functions in the system that depend on $\dot{x}_1, \dot{y}_1, \dot{x}_2, \dot{y}_2, \dot{x}_3, \dot{y}_3$, i.e. $F^8, \ldots, F^{13}$, with respect to these variables. Once this behaviour is realised, the last six equations break up with respect to these variables giving rise to a new system of differential equations ([11], appendix d).

A lot of equations in this new system are solved by the procedure of solutions, described in the previous section. To solve the remaining equations we are forced to insert the formula (4.4) for R. Though the system becomes bigger it does not 'explode' and allows us to solve it. We end up with the symmetries of the Euler–Lagrange equations ([11], appendix e). The structure of the Lie algebra generated by these symmetries is given in the appendix.

The following six variational symmetries have been used by Ruijgrok [1] to solve the three-body problem

- $\dfrac{\partial}{\partial x_1} + \dfrac{\partial}{\partial x_2} + \dfrac{\partial}{\partial x_3}$
- $\dfrac{\partial}{\partial y_1} + \dfrac{\partial}{\partial y_2} + \dfrac{\partial}{\partial y_3}$
- $\dfrac{\partial}{\partial t}$
- $$(2y_1 - y_2 - y_3)\frac{\partial}{\partial x_1} + (2y_2 - y_3 - y_1)\frac{\partial}{\partial x_2} + (2y_3 - y_1 - y_2)\frac{\partial}{\partial x_3} \qquad (4.10)$$
  $$- (2x_1 - x_2 - x_3)\frac{\partial}{\partial y_1} - (2x_2 - x_3 - x_1)\frac{\partial}{\partial y_2} - (2x_3 - x_1 - x_2)\frac{\partial}{\partial y_3}$$
- $\dfrac{\partial}{\partial x_1} - \dfrac{1}{2}\dfrac{\partial}{\partial x_2} - \dfrac{1}{2}\dfrac{\partial}{\partial x_3} - \dfrac{1}{2}\sqrt{3}\dfrac{\partial}{\partial y_2} + \dfrac{1}{2}\sqrt{3}\dfrac{\partial}{\partial y_3}$
- $\dfrac{1}{2}\sqrt{3}\dfrac{\partial}{\partial x_2} - \dfrac{1}{2}\sqrt{3}\dfrac{\partial}{\partial x_3} + \dfrac{\partial}{\partial y_1} - \dfrac{1}{2}\dfrac{\partial}{\partial y_2} - \dfrac{1}{2}\dfrac{\partial}{\partial y_3}$.

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Appendix

The structure of the Lie algebra of symmetries is more clear if we use the transformation introduced by Ruijgrok [1]

$$\begin{aligned}
\boldsymbol{Z} &= \tfrac{1}{3}(\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3) \\
\boldsymbol{C} &= \tfrac{1}{3}(\boldsymbol{r}_1 + M\boldsymbol{r}_2 + M^2\boldsymbol{r}_3) \qquad (A1) \\
R\begin{pmatrix}\cos\phi \\ \sin\phi\end{pmatrix} &= \tfrac{1}{3}(\boldsymbol{r}_1 + M^2\boldsymbol{r}_2 + M\boldsymbol{r}_3)
\end{aligned}$$

where M is the rotation over 120°. Z is the centre of mass, ϕ is drawn in figure 1.1 and R is the same as given by formula (1.2).

The transformed Lie algebra of symmetries is generated by the following symmetries

$$\begin{array}{llll}
S_1 = \partial_\phi & S_2 = t\partial_t + 2R\partial_R & S_3 = \partial_{C_y} & S_4 = t\partial_{C_y} \\
S_5 = C_y\partial_{C_y} & S_6 = C_x\partial_{C_y} & S_7 = Z_y\partial_{C_y} & S_8 = Z_x\partial_{C_y} \\
S_9 = \partial_{C_x} & S_{10} = t\partial_{C_x} & S_{11} = C_y\partial_{C_x} & S_{12} = C_x\partial_{C_x} \\
S_{13} = Z_y\partial_{C_x} & S_{14} = Z_x\partial_{C_y} & S_{15} = \partial_{Z_y} & S_{16} = t\partial_{Z_y} \\
S_{17} = C_y\partial_{Z_y} & S_{18} = C_x\partial_{Z_y} & S_{19} = Z_y\partial_{Z_y} & S_{20} = Z_x\partial_{Z_y} \\
S_{21} = \partial_t & S_{22} = \partial_{Z_x} & S_{23} = t\partial_{Z_x} & S_{24} = Z_x\partial_{Z_x} \\
S_{25} = C_y\partial_{Z_x} & S_{26} = C_x\partial_{Z_x} & S_{27} = Z_y\partial_{Z_x} &
\end{array}$$

We will denote the commutator between S_i and S_j by $[i,j]$. The non-zero commutator relations are

$$\begin{array}{llll}
[2,4] = S_4 & [2,10] = S_{10} & [2,16] = S_{16} & [2,21] = -S_{21} \\
[2,23] = S_{23} & & & \\
[3,5] = S_3 & [3,11] = S_9 & [3,17] = S_{15} & [3,25] = S_{22} \\
[4,5] = S_4 & [4,11] = S_{10} & [4,17] = S_{16} & [4,21] = -S_3 \\
[4,25] = S_{23} & & & \\
[5,6] = -S_6 & [5,7] = -S_7 & [5,8] = -S_8 & [5,11] = S_{11} \\
[5,17] = S_{17} & [5,25] = S_{25} & & \\
[6,9] = -S_3 & [6,10] = -S_4 & [6,11] = S_{12} - S_5 & [6,12] = -S_6 \\
[6,13] = -S_7 & [6,14] = -S_8 & [6,17] = S_{18} & [6,25] = S_{26} \\
[7,11] = S_{13} & [7,15] = -S_3 & [7,16] = -S_4 & [7,17] = S_{19} - S_5 \\
[7,18] = -S_6 & [7,19] = -S_7 & [7,20] = -S_8 & [7,25] = S_{27} \\
[8,11] = S_{14} & [8,17] = S_{20} & [8,22] = -S_3 & [8,23] = -S_4 \\
[8,24] = -S_8 & [8,25] = S_{24} - S_5 & [8,26] = -S_6 & [8,27] = -S_7 \\
[9,12] = S_9 & [9,18] = S_{15} & [9,26] = S_{22} & \\
[10,12] = S_{10} & [10,18] = S_{16} & [10,21] = -S_9 & [10,26] = S_{23} \\
[11,12] = S_{11} & [11,18] = S_{17} & [11,26] = S_{25} & \\
[12,13] = -S_{13} & [12,14] = -S_{14} & [12,18] = S_{18} & [12,26] = S_{26} \\
[13,15] = -S_9 & [13,16] = -S_{10} & [13,17] = -S_{11} & [13,18] = S_{19} - S_{12} \\
[13,19] = -S_{13} & [13,20] = -S_{14} & [13,26] = S_{27} & \\
[14,18] = S_{20} & [14,22] = -S_9 & [14,23] = -S_{10} & [14,24] = -S_{14} \\
[14,25] = -S_{11} & [14,26] = S_{24} - S_{12} & [14,27] = -S_{13} & \\
[15,19] = S_{15} & [15,27] = S_{22} & & \\
[16,19] = S_{16} & [16,21] = -S_{15} & [16,27] = S_{23} & \\
[17,19] = S_{17} & [17,27] = S_{25} & & \\
[18,19] = S_{18} & [18,27] = S_{26} & &
\end{array}$$

$[19,20] = -S_{20}$ $[19,27] = S_{27}$
$[20,22] = -S_{15}$ $[20,23] = -S_{16}$ $[20,24] = -S_{20}$ $[20,25] = -S_{17}$
$[20,26] = -S_{18}$ $[20,27] = S_{24} - S_{19}$
$[21,23] = S_{22}$
$[22,24] = S_{22}$
$[23,24] = S_{23}$
$[24,25] = -S_{25}$ $[24,26] = -S_{26}$ $[24,27] = -S_{27}$

The radical of this Lie algebra is generated by S_1, S_2, S_3, S_4, S_9, S_{10}, S_{15}, S_{16}, S_{20}, S_{21}, S_{22} and $S_5 + S_{12} + S_{19} + S_{24}$. The resulting quotient algebra is classified as A_3, i.e. the Lie algebra of SL(4).

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